Math 3280

Review Problems for Exam I

This is not an exhaustive list of all possible types of problems. See the class notes, the textbook and homework for additional problems.

1. Solve the IVP $x' + 2x = t\,e^{-2t}\,,\ x(1) = 0$.

2. Find the explicit solution of the IVP $\dfrac{dx}{dt} = \dfrac{3t^2 + 4t + 2}{2(x-1)}\,,\ x(0) = 2$.

3. Find the implicit solution of $\dfrac{dx}{dt} = \dfrac{x\cos t + 2\,t\,e^x}{1 - \sin t - t^2\,e^x}$.

4. Find the explicit solution $x = x(t)$ of $(t^2 + 3\,t\,x + x^2)\,dt - t^2\,dx = 0$. Assume $t > 0$.

5. Consider $x'(t) = x - \sqrt[3]{4\,x}$. Find its equilibrium solutions. Classify the equilibrium solutions and draw its phase line.

6. Consider $x'(t) = x^2(1 - x^2)$. Find its critical points and classify them using the linearization theorem. Draw its phase line.

7. Consider $x'(t) = x^4 - 2x^3 + x^2$. Find its equilibrium points and classify them. Determine the intervals, (initial) x-values, on which the solution $x(t)$ would be increasing or decreasing. Determine the intervals, x-values, on which the solution $x(t)$ would be concave up or down. Use these information to graph several representative flows (solutions) in the tx-plane.

8. Solve the IVP $6x''(t) = 5x'(t) - x(t)\,,\ x(0) = 4\,,\ x'(0) = 0$ by first converting it to a first order planar system.

9. Consider the nonlinear system

$$\frac{dx}{dt} = x\,y + 1$$
$$\frac{dy}{dt} = x^2 + 2$$

   Find all places in the xy-plane that its solution trajectories have (a) Horizontal tangent lines, (b) Vertical tangent lines, or (c) Tangent lines with slope $m = 2$, in any.

10. Solve

$$\frac{dx}{dt} = -x\,y + e^{t^2} \qquad (x(1),\,y(1)) = \left(\frac{e}{2},\,1\right)$$
$$\frac{dy}{dt} = -\frac{y}{t}$$

   Assume $t > 0$.

(Partial) Answers to the Review Problems for Exam I Continued

17. $\lambda = \lambda_1 = \lambda_2 = -1\,, V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $X(t) = c\,e^{-t}\,V$ for any vector V. All solutions are on lines through origin. The critical point (0, 0) is a sink.

18. $\lambda = i\,, V = \begin{pmatrix} 3-i \\ 5 \end{pmatrix}$; $X(t) = c_1 \begin{pmatrix} 3\cos t + \sin t \\ 5\cos t \end{pmatrix} + c_2 \begin{pmatrix} 3\sin t - \cos t \\ 5\sin t \end{pmatrix}$. The critical point (0, 0) is a center. Solution curves have clockwise rotation.

19. $\lambda = 1 + 2i\,, V = \begin{pmatrix} -i \\ 1 \end{pmatrix}$; $X(t) = c_1 \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} -e^t \cos 2t \\ e^t \sin 2t \end{pmatrix}$. The critical point (0, 0) is a spiral source. Solution curves have clockwise rotation.

20. $\lambda_1 = -2\,, V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -1\,, V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $X(t) = c_1\,e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\,e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The lines $y = x$ and $y = 2x$ are the straight-line solutions (stable manifolds); The critical point (0, 0) is a sink.

11. Consider the nonlinear system

$$\frac{dx}{dt} = 3y^2(x^2+1)$$
$$\frac{dy}{dt} = 2x$$

Solve it to find y as a function of x by considering $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

12. Find the solution of $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ with initial values $x(0) = 2$ and $y(0) = 1$

In the following problems, consider the system $X'(t) = A\,X$, with the given matrix A, and do the following.

(a) Find its general solution.

(b) Find its critical point(s) and classify them, if possible.

(c) Discuss information useful for drawing its representative trajectories, for example,

- Straight line (invariant) solutions.
- Slopes of tangent lines to the solution curves.
- Direction field or the direction of the increasing t value of the solution curves.
- Converting the solution to rectangular coordinates.

(d) Draw several representative trajectories.

Note: For these review problems, you may use Mathematica to find the eigenvalues and the corresponding eigenvectors.

13. $A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

14. $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

15. $A = \begin{pmatrix} -3 & -1 \\ 4 & -1 \end{pmatrix}$

16. $A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$

17. $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

18. $A = \begin{pmatrix} -3 & 2 \\ -5 & 3 \end{pmatrix}$

19. $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

20. $A = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}$

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(Partial) Answers to the Review Problems for Exam I

1. $x(t) = \dfrac{1}{2}\,(t^2 - 1)\, e^{-2t}$

2. $x(t) = 1 + \sqrt{t^3 + 2\,t^2 + 2\,t + 1}$

3. $x - x\sin t - t^2\, e^x = c$

4. $x(t) = \dfrac{(1-c)\,t - t\,\ln t}{c + \ln t}$

5. $x = -2$, unstable; $x = 0$, stable; $x = 2$, unstable

6. $x = -1$, unstable; $x = 0$, semistable; $x = 1$, stable

7. $x = 0$ and $x = 1$, both semistable; x is increasing for $x < 0$, $0 < x < 1$, and $x > 1$; x is concave downward for $x < 0$ and $\frac{1}{2} < x < 1$; x is concave upward for $0 < x < \frac{1}{2}$ and $x > 1$

8. $x = 12\, e^{\frac{1}{3}t} - 8\, e^{\frac{1}{2}t}$

9. No horizontal tangent lines; Vertical tangent lines on the curve $y = -\dfrac{1}{x}$; Tangent lines with slope $m = 2$ on the lines $x = 0$ and $y = \dfrac{x}{2}$

10. $x(t) = \dfrac{e^{t^2}}{2\,t}\,,\ y(t) = \dfrac{1}{t}$

11. $y = -\sqrt[3]{c + \ln(x^2+1)}$

12. $x(t) = (t+2)\, e^{2\,t}\,,\ y(t) = e^{2\,t}$

13. $\lambda_1 = -1\,, V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = 2\,, V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$; $X(t) = c_1\, e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\, e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The lines $y = 2x$ and $y = -x$ are the straight-line solutions (stable and unstable manifolds); The critical point $(0,\, 0)$ is a saddle point.

14. $\lambda_1 = 1\,, V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = 2\,, V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $X(t) = c_1\, e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\, e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The lines $y = 2x$ and $y = x$ are the straight-line solutions (unstable manifolds); The critical point $(0,\, 0)$ is a source.

15. $\lambda = \lambda_1 = \lambda_2 = 1\,, V = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $X(t) = c_1\, e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\, e^{t} \left(t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right)$; The lines $y = 2x$ is the straight-line solution (unstable manifolds; The critical point $(0,\, 0)$ is a source; Solutions curves try to spiral about the critical point, but the unstable manifold gets in the way.

16. $\lambda_1 = -1\,, V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = 0\,, V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $X(t) = c_1\, e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. All points on the line $y = x$ are critical points. All other solutions are lines with slope $m = 2$ that move toward the line $y = x$.